

ON THE VOLUME AND THE NUMBER OF LATTICE POINTS OF SOME SEMIALGEBRAIC SETS

HA HUY VUI * AND TRAN GIA LOC †

* *Institute of mathematics, Vietnam Academy of Sciences, Hanoi, Vietnam*
18, Hoang Quoc Viet, Cau Giay, Hanoi

† *Teacher Training College of Dalat, 29 Yersin road, Dalat, Vietnam*

ABSTRACT. Let $f = (f_1, \dots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a polynomial map; $G^f(r) = \{x \in \mathbb{R}^n : |f_i(x)| \leq r, i = 1, \dots, m\}$. We show that if f satisfies the Mikhailov - Gindikin condition then

- (i) Volume $G^f(r) \asymp r^\theta (\ln r)^k$
- (ii) Card $\left(G^f(r) \cap \overset{o}{\mathbb{Z}^n}\right) \asymp r^{\theta'} (\ln r)^{k'}, \text{ as } r \rightarrow \infty,$

where the exponents θ, k, θ', k' are determined explicitly in terms of the Newton polyhedra of f .

Moreover, the polynomial maps satisfy the Mikhailov - Gindikin condition form an open subset of the set of polynomial maps having the same Newton polyhedron.

Keywords and phrases: Newton polyhedron, the Mikhailov - Gindikin condition, sublevel set, lattice points.

1. INTRODUCTION

The study of the asymptotic behavior of the volume of sublevel sets and the number of lattice points has attracted a lot of researchers and has found many important applications. In the middle 1970s, A.N. Varchenko and V.A. Vasiliev used Newton polyhedra to study the asymptotic behavior of the volume of sublevel sets and the integrals of real analytic functions in the degenerate situation ([21], [22], [23]). In particular, sharp estimates for the volume and the integrals were obtained in terms of Newton polyhedra for certain classes of the functions with an isolated minimum at zero.

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a polynomial map. For $r > 0$, put

$$G^f(r) = \{x \in \mathbb{R}^n : |f_i(x)| \leq r, i = 1, \dots, m\}, \quad Z^f(r) = G^f(r) \cap \overset{o}{\mathbb{Z}^n},$$

where $\overset{o}{\mathbb{Z}^n} = \{(a_1, \dots, a_n) \in \mathbb{Z}^n : a_i \neq 0, i = 1, \dots, n\}$. Let $|G^f(r)|$ and Card $Z^f(r)$ be correspondingly the volume of $G^f(r)$ and the cardinal of $Z^f(r)$.

In this paper, we are interested in computing explicitly the exponents arising in the asymptotic formulas for $|G^f(r)|$ and Card $Z^f(r)$, as $r \rightarrow \infty$.

In the case of $m = 1$, the asymptotic behavior of $|G^f(r)|$ plays an important role in many problems of the theory of pseudo-differential operators.

The asymptotic behavior of the volume of the set $\{x \in U : |f(x)| < r\}$, as $r \rightarrow 0$, where U is a small enough neighborhood of a singularity point, concerns the oscillatory integral operators and the scalar oscillatory integrals (see [5], [8], [9], [11], [12], [13], [16], [18], [9],

2010 *Mathematics Subject Classification.* 14B05, 32S20, 34E05, 11H06, 51M20, 52A38, 52A23.

Key words and phrases. Newton polyhedron, the Mikhailov - Gindikin condition, sublevel set, lattice points.

[19]).

The asymptotic behavior of the volume of the set $\{x \in \mathbb{R}^n : |f(x)| < r\}$, as $r \rightarrow \infty$, is used in [20] to estimate the number of eigenvalues of the Schrödinger operator in \mathbb{R}^n .

In [4], the asymptotic behavior of $\text{Card } Z^f(r)$, where f is a *monomial map* was computed and applied in the approximation theory.

For the set $G^f(r)$, the following problems arise naturally.

- (i) When are the qualities $|G^f(r)|$ and $\text{Card } Z^f(r)$ finite.
- (ii) If they are finite, how to compute the exponents arising in the asymptotic formulas for these qualities?

If f is an arbitrary polynomial map then it is very difficult to provide satisfactory answers to these questions, even for the case $n = 2$ and $m = 1$. However, if the application f satisfies the so called Mikhailov - Gindikin condition, then we can give complete answers to these problems.

2. STATEMENT OF RESULTS

For a polynomial $\varphi(x) = \sum a_\alpha x^\alpha \in \mathbb{R}[x_1, \dots, x_n]$, we call the support of φ the following set

$$\text{supp}(\varphi) := \{\alpha \in (\mathbb{N} \cup \{0\})^n : a_\alpha \neq 0\}.$$

Let $f = (f_1, \dots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Put $\Gamma(f) = \text{convex} \left(\bigcup_{i=1}^m \text{supp}(f_i) \right)$, the convex hull of the set $\bigcup_{i=1}^m \text{supp}(f_i)$. We call $\Gamma(f)$ the *Newton polyhedron* of f .

Let $f_i(x) = \sum a_\alpha^i x^\alpha$ and Δ be a face of $\Gamma(f)$. We put

$$f_{i\Delta}(x) = \sum_{\alpha \in \Delta} a_\alpha^i x^\alpha.$$

Definition 2.1. We say that f satisfies the Mikhailov-Gindikin condition if for any face $\Delta \subset \Gamma(f)$, we have

$$\max |f_{i\Delta}(x)| \neq 0, \quad i = 1, \dots, m;$$

in $(\mathbb{R} \setminus \{0\})^n$.

We denote by $\text{cone}\Gamma(f)$ the cone generated by $\Gamma(f)$,

$$\text{cone}\Gamma(f) = \{y : y = \lambda x \text{ for } \lambda \geq 0 \text{ and } x \in \Gamma(f)\},$$

and by $\Delta^+(d)$ the diagonal of the positive orthant in \mathbb{R}^n ,

$$\Delta^+(d) = \{(d_1, \dots, d_n) \in \mathbb{R}_+^n : d_1 = \dots = d_n\}.$$

Let $D_\infty\Gamma(f)$ be the furthest point from the origin in the intersections of the diagonal $\Delta^+(d)$ and $\Gamma(f)$, and Λ_∞ be the face of smallest dimension of $\Gamma(f)$, having $D_\infty\Gamma(f)$ as its interior point.

We denote by $k = \dim\Lambda_\infty$, $D_\infty\Gamma(f) = (d_\infty, \dots, d_\infty)$, $\theta = \frac{1}{d_\infty}$ and $v_n = (1, \dots, 1)$.

The notation $g \asymp h$ means that there exists positive constants K_1, K_2 such that

$$K_1 h \leq g \leq K_2 h.$$

Theorem 2.2. Let $f = (f_1, \dots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a polynomial map satisfying the Mikhailov-Gindikin condition. Then we have

- (i) $|G^f(r)| < \infty$, for any $r > 0$, if and only if the vector $(1, \dots, 1)$ belongs to the interior of $\text{cone}\Gamma(f)$.
- (ii) If $|G^f(r)| < \infty$, then we have

$$|G^f(r)| \asymp r^\theta \ln^{n-k-1} r, \text{ as } r \rightarrow \infty.$$

Next, we construct the so called *complete Newton polyhedron* of f .

For $\alpha, \beta \in \mathbb{R}^n$, we shall write $\alpha \preceq \beta$, if $\alpha_j \leq \beta_j$ for all $j = 1, \dots, n$.

Definition 2.3. We call the complete Newton polyhedron of f , the polyhedron $\tilde{\Gamma}(f)$ obtained from $\Gamma(f)$ by adding all the $\alpha \in \overline{\mathbb{R}_+^n}$ for which there exists $\beta \in \Gamma(f)$, s.t. $\alpha \preceq \beta$.

We denote by $D_\infty \tilde{\Gamma}(f)$ the furthest point from the origin in the intersections of $\Delta^+(d)$ and $\tilde{\Gamma}(f)$. Put $D_\infty \tilde{\Gamma}(f) = (\tilde{d}_\infty, \dots, \tilde{d}_\infty)$ and $\theta' = \frac{1}{\tilde{d}_\infty}$. Let $\tilde{\Lambda}_\infty$ be the face having smallest dimension of $\tilde{\Gamma}(f)$ that contains the point $D_\infty \tilde{\Gamma}(f)$ in its interior. Put $k' = \dim \tilde{\Lambda}_\infty$.

Theorem 2.4. Let $f = (f_1, \dots, f_n) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a polynomial map satisfying the Mikhailov-Gindikin condition. Then we have

- (i) $\text{Card } Z^f(r) < \infty$ for all positive real numbers r , if and only if $\text{cone}\Gamma(f) \cap \overset{\circ}{\mathbb{R}^n} \neq \emptyset$.
- (ii) Moreover, if $\text{Card } Z^f(r) < \infty$, we have

$$\text{Card } Z^f(r) \asymp r^{\theta'} \ln^{n-k'-1} r \text{ as } r \rightarrow \infty.$$

REMARK 2.5.

- (i) It follows from Theorems 2.2 and 2.4 that, under the Mikhailov - Gindikin condition, the equalities $\theta = \theta'$, $k = k'$ hold if and only if $\Lambda = \tilde{\Lambda}$, i.e Λ is a common face of $\Gamma(f)$ and $\tilde{\Gamma}(f)$.
- (ii) If f is a monomial map, the exponents in the asymptotic formulas for $G^f(r)$ and $\text{Card } Z^f(r)$ were computed by Dinh Dung in [4]. Note that this author has stated his result in terms of some linear programming problems and did not make use of Newton polyhedra.

Let Γ be a convex polytope in \mathbb{R}^n . Assume that all the vertice of Γ belong to $(\mathbb{N} \cup \{0\})^n$.

We define

$$\mathcal{M}_\Gamma := \left\{ f : \mathbb{R}^n \rightarrow \mathbb{R}^m : \bigcup_{i=1}^m \text{supp}(f_i) \subset \Gamma \right\}, \quad \mathcal{N}_\Gamma := \{ f : \mathbb{R}^n \rightarrow \mathbb{R}^m : \Gamma(f) = \Gamma \},$$

$$\mathcal{D}_\Gamma := \{ f : \mathbb{R}^n \rightarrow \mathbb{R}^m : \Gamma(f) = \Gamma, \text{ and } f \text{ satisfies the Mikhailov-Gindikin condition} \}.$$

By the lexicographic ordering in the set of monomials, we can identify \mathcal{M}_Γ with a finite dimensional space over \mathbb{R} , and \mathcal{N}_Γ and \mathcal{D}_Γ with subsets in this space.

Theorem 2.6. With the notations above, \mathcal{D}_Γ is an open subset in \mathcal{N}_Γ , and, consequently, it is an open set in the space \mathcal{M}_Γ .

3. PROOFS

Theorem 2.2 and Theorem 2.4 are direct consequences of two following facts

- (i) Two-side estimation for polynomial functions satisfying the Mikhailov - Gindikin condition.

- (ii) Asymptotic formulas for the volume and the number of lattice points in semi-algebraic sets, defined by monomial inequalities [4].

Let $\varphi(x) = \sum a_\alpha x^\alpha \in \mathbb{R}[x_1, \dots, x_n]$, $\Gamma(\varphi)$ be the Newton polyhedron of φ and $V(\varphi)$ be the set of vertices of $\Gamma(\varphi)$.

$$\text{Put } N_\varphi(x) = \sum_{\alpha \in V(\varphi)} |x^\alpha|.$$

Theorem 3.1. (see [6, p. 204]) *Two conditions are equivalent*

- (i) *There is $c > 0$ and $\rho > 0$ such that*

$$cN_\varphi(x) \leq |\varphi(x)|, \quad x \in \mathbb{R}^n, |x| > \rho.$$

- (ii) *For any face $\Delta \subset \Gamma(\varphi)$, and $x \in (\mathbb{R} \setminus \{0\})^n$, $|x| > \rho$, we have*

$$\varphi(x) \neq 0, \quad \text{and} \quad \varphi_\Delta(x) \neq 0.$$

REMARK 3.2. It follows from the theorem 3.1 and from ([6, Lemma 1.1]) that if φ satisfies condition (ii), then exist positive constants c_1 , c_2 and ρ such that

$$c_1 N_\varphi(x) \leq |\varphi(x)| \leq c_2 N_\varphi(x), \quad x \in \mathbb{R}^n, |x| > \rho.$$

Now, let us consider the system of monomials

$$\{x^{\alpha^1}, \dots, x^{\alpha^s}\}, \quad \alpha^i \in (\mathbb{N} \cup \{0\})^n, \quad i = 1, \dots, s.$$

For $r > 0$, put

$$G^\alpha(r) = \left\{ x \in \mathbb{R}^n : |x|^{\alpha^i} \leq r, \quad i = 1, \dots, s \right\}.$$

In [4], Dinh Dung computed the first term in asymptotic formulas for volume of $G^\alpha(r)$ and for the number of lattice points in $Z^\alpha(r) = G^\alpha(r) \cap \overset{o}{\mathbb{Z}^n}$. We now recall his result.

Consider the following linear programming problem

$$(3.1) \quad \begin{cases} x_1 + \dots + x_n \rightarrow \sup; \\ \langle x, \alpha^i \rangle \leq 1, \quad i = 1, \dots, s \\ x \in \mathbb{R}^n. \end{cases}$$

Let θ and $M(\alpha)$ be correspondingly the optimal value and the solution set of this problem. Put $p := \dim M(\alpha)$ and $\Gamma(\alpha) := \text{conv} \{ \alpha^1, \dots, \alpha^s \}$, and let $C(\alpha)$ be the cone generated by $\Gamma(\alpha)$.

Theorem 3.3. ([4, Theorem 1]) *The volume of $G^\alpha(r)$ is finite for all $r > 0$ if and only if the vector $(1, \dots, 1)$ belongs to the interior of $C(\alpha)$. Moreover, if volume of $G^\alpha(r)$ is finite for all $r > 0$, then*

$$\text{volume of } G^\alpha(r) \asymp r^\theta \ln^p r.$$

Next, consider the linear programming problem

$$(3.2) \quad \begin{cases} x_1 + \dots + x_n \rightarrow \sup; \\ \langle x, \alpha^i \rangle \leq 1, \quad i = 1, \dots, s \\ x \in \mathbb{R}_+^n. \end{cases}$$

Theorem 3.4. ([4, Theorem 2]) *Card $Z^\alpha(r)$ is finite for any $r > 0$ if and only if $C(\alpha) \cap \overset{\circ}{\mathbb{R}^n} \neq \emptyset$. Moreover, if this condition is satisfied, then*

$$\text{Card } Z^\alpha(r) \asymp r^{\theta'} \ln^{p'} r ,$$

where θ' and p' be correspondingly the optimal value and the dimension of the solution set of the linear programming problem (3.2).

Proof of Theorem 2.2

Assume that $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfies the Mikhailov-Gindikin condition. We put $N_f(x) := \sum_{\alpha \in V(f)} |x|^\alpha$, where $V(f)$ is the set of vertices of $\Gamma(f)$.

Lemma 3.5. *If f satisfies the Mikhailov-Gindikin condition then there exist positive constants c_1 , c_2 and ρ such that*

$$c_1 N_f(x) \leq \max |f_i(x)| \leq c_2 N_f(x)$$

for all $x \in \mathbb{R}^n$, $|x| > \rho$.

PROOF. Let $F = \sum_{i=1}^m f_i^2$ and $\Gamma(F)$ be the Newton polyhedron of F . It is not difficult to see that if $V(f) = \{\alpha^1, \dots, \alpha^k\}$ is the set of vertices of $\Gamma(f)$ then the set $V(F) = \{2\alpha^1, \dots, 2\alpha^k\}$ is that of $\Gamma(F)$. In consequence, for every face Δ' of $\Gamma(F)$, there exists a unique face Δ of $\Gamma(f)$ such that $\Delta' = 2\Delta$.

Claim 3.6. *Let Δ' be a face of $\Gamma(F)$ and Δ be that of $\Gamma(f)$ such that $\Delta' = 2\Delta$. Then*

$$F_{\Delta'}(x) = \sum_{i=1}^m f_{i\Delta}^2(x) .$$

PROOF.

We begin with a description of a face of a polyhedron in \mathbb{R}^n . Let Γ be a polyhedron in \mathbb{R}^n , $\dim \Gamma = n$ and Δ be its face. Then there exists $q \in \mathbb{R}^n$ such that the restriction of $\langle x, q \rangle$ on Γ attains its maximum value at x if and only if $x \in \Delta$.

In fact, if $\dim \Delta = n - 1$ then q is a normal vector of the hyperplane containing Δ , and q is determined uniquely within a positive factor. If $\dim \Delta < n - 1$, then Δ lies on the boundaries of some faces of dimension $n - 1$, say $\Delta_1, \dots, \Delta_l$, where

$$\Delta_i = \{x \in \Gamma : \langle x, q_i \rangle = d(q_i)\}$$

and $d(q_i) = \sup_{x \in \Gamma} \langle x, q_i \rangle$. Then Δ can be represented by

$$\Delta = \{x \in \Gamma : \langle x, q \rangle = d(q)\}$$

with $q = \sum_{i=1}^l t_i q_i$, $\sum_{i=1}^l t_i = 1$ and $t_i > 0$, $i = 1, \dots, l$.

Now, assume that

$$\Delta = \{x \in \Gamma(f) : \langle x, q \rangle = d(q)\} \quad \text{and} \quad \Delta' = \{x \in \Gamma(F) : \langle x, q \rangle = 2d(q)\} .$$

We write $f_i(x)$ in the form

$$f_i(x) = h_i(x) + g_i(x)$$

where $h_i(x) = f_{i\Delta}(x)$. In the sum

$$f_i^2(x) = h_i^2(x) + 2h_i(x)g_i(x) + g_i^2(x)$$

every monomial x^α satisfying the condition $\langle \alpha, q \rangle = 2d(q)$, can occur only in the first summand. Therefore

$$F_{\Delta'}(x) = \sum_{i=1}^m h_i^2(x) = \sum_{i=1}^m f_{i_{\Delta}}^2(x).$$

As consequence of this claim, since f satisfies the Mikhailov - Gindikin condition, F satisfies this condition too.

By Theorem 3.1, there exist $c > 0$, $c' > 0$, $\rho > 0$ such that

$$|x| > \rho \quad \Rightarrow \quad c \sum_{2\alpha^i \in V(F)} |x^{2\alpha^i}| \leq |F(x)| \leq c' \sum_{2\alpha^i \in V(F)} |x^{2\alpha^i}|$$

for all $x \in \mathbb{R}^n$, where $V(F)$ is the set of vertices of $\Gamma(F)$. And therefore, there exist positive constants c_1 , c_2 and ρ_1 such that

$$c_1 \sum_{\alpha \in V(f)} |x^\alpha| \leq \max |f_i(x)| \leq c_2 \sum_{\alpha \in V(f)} |x^\alpha|,$$

for $|x| \geq \rho_1$. □

Put

$$\mathcal{A}(r) := \{x \in \mathbb{R}^n : |x|^\kappa \leq r, \kappa \in V(f)\}, \quad \mathcal{B}(r) := \{x \in \mathbb{R}^n : \max |f_i(x)| \leq r\}.$$

Now, it follows from Lemma 3.5 that there exist constants ρ_1 and ρ_2 such that

$$(3.3) \quad \|x\| \geq \rho \quad \Rightarrow \quad \rho_1 |\mathcal{A}(r)| \leq |\mathcal{B}(r)| \leq \rho_2 |\mathcal{A}(r)|.$$

Since

$$|\mathcal{B}(r)| = |\{x \in \mathbb{R}^n : \max |f_i(x)| \leq r\}|$$

and

$$|\{x \in \mathbb{R}^n : \|x\| \leq \rho\}| \leq |\{x \in \mathbb{R}^n : \max |f_i(x)| \leq r\}|$$

then

$$|\mathcal{B}(r)| \asymp |\{x \in \mathbb{R}^n : \max |f_i(x)| \leq r\}|, \quad r \rightarrow \infty.$$

Thus, by (3.3), the proof of Theorem 2.2 is reduced to the problem of computing the exponents in the asymptotic formula of $|\mathcal{A}(r)|$, as $r \rightarrow \infty$. For this monomial case, the problem is solved already in [4].

Using Theorem 3.3, we have

- (i) $|\mathcal{A}(r)| < \infty$ for any $r > 0$ if and only if $v_n \in \text{int}(\text{cone}V(f))$.
- (ii) If $|\mathcal{A}(r)| < \infty$ then $|\mathcal{A}(r)| \asymp r^{\tilde{\theta}} \ln^{\tilde{k}} r$, where $\tilde{\theta}$ is the optimal value and \tilde{k} is the dimension of the solution set of the following linear programming problem

$$(3.4) \quad \begin{cases} x_1 + \dots + x_n \rightarrow \sup; \\ \langle x, \alpha^i \rangle \leq 1, \quad \alpha^i \in V(f) = \{\alpha_1, \dots, \alpha_s\} \\ x \in \mathbb{R}^n, \end{cases}$$

To finish the proof of Theorem 2.2, it rest to prove that $\tilde{\theta} = \theta$, and $\tilde{k} = n - k - 1$, where the exponents θ and k are determined in the statement of Theorem 2.2.

We write the linear programming problem above in the form

$$(3.5) \quad \begin{cases} \max \{x_1 + \dots + x_n\}, \\ \alpha_1^1 x_1 + \dots + \alpha_n^1 x_n \leq 1 \\ \dots \\ \alpha_1^s x_1 + \dots + \alpha_n^s x_n \leq 1 \\ (x_1, \dots, x_n) \in \mathbb{R}^n, \end{cases}$$

where $\alpha^i = (\alpha_1^i, \dots, \alpha_n^i)$, $i = 1, \dots, s$.

Let us consider the dual problem

$$(3.6) \quad \begin{cases} \min \{u_1 + \dots + u_s\}, \\ \alpha_1^1 u_1 + \dots + \alpha_n^1 u_s = 1 \\ \dots \\ \alpha_1^s u_1 + \dots + \alpha_n^s u_s = 1 \\ u_i \geq 0, \quad i = 1, \dots, s. \end{cases}$$

The system of linear equations in (3.6) can be rewritten

$$(3.7) \quad \left(\frac{u_1}{\sum_{i=1}^s u_i} \right) \alpha^1 + \dots + \left(\frac{u_s}{\sum_{i=1}^s u_i} \right) \alpha^s = \left(\frac{1}{\sum_{i=1}^s u_i}, \dots, \frac{1}{\sum_{i=1}^s u_i} \right).$$

The point in the left-hand side of (3.7) belongs to $\text{conv}\{\alpha^1, \dots, \alpha^s\}$, and the right-hand side is a point that belongs to $\Delta^+(d)$. On the other hand, $\frac{1}{\sum_{i=1}^s u_i}$ achieves the maximum value when $\sum_{i=1}^s u_i$ reaches the minimum value. Thus $\sum_{i=1}^s u_i$ achieves the minimum value at the point $D_\infty \Gamma(f) = (d_\infty, \dots, d_\infty)$ and $\tilde{\theta} = \frac{1}{d_\infty} = \theta$.

Put $P := \{x \in \mathbb{R}^n : \langle x, \alpha^i \rangle \leq 1, \quad i = 1, \dots, s\}$ and let P^* be the polar set of P , i.e.

$$P^* = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \quad \forall x \in P\}.$$

By ([3], Theorem 9.1, p.57), we have

$$P^* := \text{conv}\{O, \alpha^1, \dots, \alpha^s\}.$$

According to The Bipolar Theorem ([2]), $(P^*)^* = P$.

Let Λ_{\max} be the solution set of the problem (3.4). Then Λ_{\max} is a face of P . Put

$$\Lambda_\infty = \{y \in P^* : \langle x, y \rangle = 1, \quad \forall x \in \Lambda_{\max}\}.$$

Then, Λ_∞ is a face of P^* and Λ_∞^* , the polar set of Λ_∞ , is equal to Λ_{\max} . We see that if $x \in \Lambda_{\max}$, then $x_1 + \dots + x_n = \theta$. Therefore $\langle D_\infty \Gamma(f), x \rangle = \frac{1}{\theta} (x_1 + \dots + x_n) = 1$, hence $D_\infty \Gamma(f) \in \Lambda_\infty$. Since Λ_∞ does not contain the origin, Λ_∞ is the face of $\Gamma(f)$ containing the point $D_\infty \Gamma(f)$.

Now, since $\Lambda_\infty = \Lambda_{\max}^*$, we have

$$\dim \Lambda_{\max} = \tilde{k} = n - \dim \Lambda_\infty - 1 = n - k - 1.$$

□

3.1. Proof of Theorem 2.4.

As in the proof of Theorem 2.2, the proof of Theorem 2.4 is reduced to the problem of computing the exponents in the asymptotic formula $\text{card } Z^{V(f)}(r)$, as $r \rightarrow \infty$, where $\text{card } Z^{V(f)}(r) = \mathcal{A}(r) \cap \overset{o}{\mathbb{Z}}^n$. Using Theorem 3.4, we have

- (i) $\text{Card } Z^{V(f)}(r) < \infty$ for any $r > 0$ if and only if $\text{cone} V(f) \cap \overset{o}{\mathbb{R}}^n \neq \emptyset$.
- (ii) If $\text{Card } Z^{V(f)}(r) < \infty$ then $\text{Card } Z^{V(f)}(r) \asymp r^{\tilde{\theta}'} \ln^{\tilde{k}'} r$, where $\tilde{\theta}'$ is the optimal value and \tilde{k}' is the dimension of the solution set of the following linear programming problem

$$(3.8) \quad \begin{cases} x_1 + \dots + x_n \rightarrow \sup ; \\ \langle x, \alpha^i \rangle \leq 1, \quad \alpha^i \in V(f) = \{\alpha_1, \dots, \alpha_s\} \\ x \in \mathbb{R}_+^n, \end{cases}$$

We will show that $\tilde{\theta}' = \theta'$, and $\tilde{k}' = n - k' - 1$, where the exponents θ' and k' are determined in the statement of Theorem 2.4.

We write the linear programming problem 3.8 in the form

$$(3.9) \quad \begin{cases} \max \{x_1 + \dots + x_n\}, \\ \alpha_1^1 x_1 + \dots + \alpha_n^1 x_n \leq 1 \\ \dots \\ \alpha_1^s x_1 + \dots + \alpha_n^s x_n \leq 1 \\ (x_1, \dots, x_n) \in \mathbb{R}^n, \quad x_j \geq 0, \quad j = 1, \dots, n, \end{cases}$$

where $\alpha^i = (\alpha_1^i, \dots, \alpha_n^i)$, $i = 1, \dots, s$.

Let us consider the dual problem

$$(3.10) \quad \begin{cases} \min \{u_1 + \dots + u_s\}, \\ \alpha_1^1 u_1 + \dots + \alpha_n^1 u_s \geq 1 \\ \dots \\ \alpha_1^s u_1 + \dots + \alpha_n^s u_s \geq 1 \\ u_i \geq 0, \quad i = 1, \dots, s. \end{cases}$$

The system of linear inequations in (3.10) can be rewritten

$$(3.11) \quad \left(\frac{u_1}{\sum_{i=1}^s u_i} \right) \alpha^1 + \dots + \left(\frac{u_s}{\sum_{i=1}^s u_i} \right) \alpha^s \geq \left(\frac{1}{\sum_{i=1}^s u_i}, \dots, \frac{1}{\sum_{i=1}^s u_i} \right).$$

Put $\gamma^l = \left(\frac{u_1}{\sum_{i=1}^s u_i} \right) \alpha^1 + \dots + \left(\frac{u_s}{\sum_{i=1}^s u_i} \right) \alpha^s$ and $\gamma^r = \left(\frac{1}{\sum_{i=1}^s u_i}, \dots, \frac{1}{\sum_{i=1}^s u_i} \right)$. Since $\gamma^l \in \Gamma(f)$ and $\gamma^r \leq \gamma^l$, we have $\gamma^r \in \tilde{\Gamma}(f)$, the complete Newton polyhedron of f .

On the other hand, $\frac{1}{\sum_{i=1}^s u_i}$ achieves the maximum value when $\sum_{i=1}^s u_i$ reaches the minimum value. It follows that $\sum_{i=1}^s u_i$ reaches the minimum value at the point $D_\infty \tilde{\Gamma}(f) = (\tilde{d}_\infty, \dots, \tilde{d}_\infty)$ and $\tilde{\theta}' = \frac{1}{\tilde{d}_\infty} = \theta'$.

Put $\tilde{P} = \{x \in \mathbb{R}_+^n : \langle x, \alpha^i \rangle \leq 1, \quad i = 1, \dots, s\}$. Then, \tilde{P} is a bounded convex polyhedron having faces which intersect the axes Ox_j at the points A_j , $j = 1, \dots, n$, and containing the

origin O . Hence,

$$\tilde{P} = \text{conv}\{O, A_1, \dots, A_n, \alpha_1, \dots, \alpha_s\}.$$

From the properties of polar sets (see [10, 24]) we have, $\tilde{P}^* = \bigcap_{\beta \in V(\tilde{P})} K(\beta, 1)$, where $K(\beta, 1) = \{x \in \mathbb{R}^n : \langle \beta, x \rangle \leq 1\}$ and $V(\tilde{P})$ is the set of the vertices of \tilde{P} . Hence,

$$\tilde{P}^* \cap \overline{\mathbb{R}_+^n} = \tilde{\Gamma}(f).$$

Let $\tilde{\Lambda}_{max}$ be the solution set of the problems (3.8). Put

$$\tilde{\Lambda}_\infty := \{y \in \tilde{P}^* : \langle x, y \rangle = 1, \text{ for all } x \in \tilde{P}\}.$$

Then, $\tilde{\Lambda}_\infty$ is a face of \tilde{P}^* and $(\tilde{\Lambda}_{max})^* = \tilde{\Lambda}_\infty$. We see that if $x \in \tilde{\Lambda}_{max}$, then $x_1 + \dots + x_n = \theta'$. Therefore $\langle D_\infty \tilde{\Gamma}(f), x \rangle = \frac{1}{\theta'}(x_1 + \dots + x_n) = 1$. Hence, $D_\infty \tilde{\Gamma}(f) \in \tilde{\Lambda}_\infty$. Since $\tilde{\Lambda}_\infty$ does not contain the origin, $\tilde{\Lambda}_\infty$ is the face of $\tilde{\Gamma}(f)$, which contains the point $D_\infty \tilde{\Gamma}(f)$ and

$$\dim \tilde{\Lambda}_{max} = \tilde{k}' = n - \dim \tilde{\Lambda}_\infty - 1 = n - k' - 1.$$

3.2. Proof of Theorem 2.6.

Put $\Omega := 2\Gamma$ and $N_\Omega := \{h \in \mathbb{R}[x_1, \dots, x_n] : \Gamma(h) = \Omega\}$. We consider the map

$$F : \mathcal{N}_\Gamma \longrightarrow N_\Omega, \quad g = (g_1, \dots, g_m) \longmapsto F_g = \sum_{i=1}^m g_i^2,$$

where $F_g(x) = \sum_{i=1}^m g_i^2(x)$. It is obvious that F is a continuous mapping. Put

$$A_\Omega := \left\{ f \in N_\Omega : \text{there exist } r > 0, c > 0 \text{ such that } \|x\| \geq r \Rightarrow f(x) \geq c \sum_{\alpha \in V(\Omega)} x^\alpha \right\}.$$

Claim 3.7. $g = (g_1, \dots, g_m) \in \mathcal{D}_\Gamma$ if and only if $F_g \in A_\Omega$.

PROOF. Let $g = (g_1, \dots, g_m) \in \mathcal{D}_\Gamma$. Then $\Gamma(g) = \Gamma$, and for any face Δ of $\Gamma(g)$, we have

$$\max |g_{i,\Delta}(x)| \neq 0 \text{ for all } x \in (\mathbb{R}^n \setminus \{0\})^n.$$

Let Δ' be a face of Ω , $\Delta' = 2\Delta$. By Claim 3.6, we have

$$F_g(x) = \sum_{i=1}^m g_i^2(x) \neq 0, \text{ and } (F_g)_{\Delta'}(x) = \sum_{i=1}^m g_{i,\Delta}^2(x) \neq 0.$$

Therefore F_g satisfies the Mikhailov - Gindikin. By Theorem 3.1, there exists $c > 0$ and $\rho > 0$ such that

$$|F_g(x)| \geq c \sum_{\alpha \in V_\Omega} |x^\alpha| = c \sum_{\alpha \in V_\Omega} x^\alpha, \text{ for all } x \in \mathbb{R}^n \text{ satisfying } |x| > \rho.$$

since all the point $\alpha \in V_\Omega$ have even coordinates.

We now prove the converse.

Let $F_g \in A_\Omega$. Then $\Gamma(F_g) = 2\Gamma(g)$ and there exist the numbers $r > 0$, $c > 0$ such that

$$\|x\| \geq r \implies F_g(x) = \sum_{i=1}^m g_i^2(x) \geq c \sum_{\alpha \in V_\Omega} x^\alpha.$$

Let Δ be a face of $\Gamma(g)$, and $\Delta' = 2\Delta$ be the corresponding face of Ω . Let $q = (\rho_1, \dots, \rho_n)$ be an interior point of the normal cone of Δ . Then

$$\Delta = \{x \in \Gamma : \langle x, q \rangle = d(q)\},$$

and $\langle x, q \rangle < d(q)$ if $x \in \Gamma \setminus \Delta$.

Take a point $x^0 \in (\mathbb{R} \setminus \{0\})^n$, we see that

$$\begin{aligned} F_g(t^{\rho_1}x_1^0, \dots, t^{\rho_n}x_n^0) &= (F_g)_{\Delta'}(t^{\rho_1}x_1^0, \dots, t^{\rho_n}x_n^0) + \text{lower order terms in } t \\ &= t^{2d(q)}(F_g)_{\Delta'}(x^0) + o(t^{2d(q)}), \quad t \rightarrow \infty. \end{aligned}$$

Hence

$$t^{2d(q)}(F_g)_{\Delta'}(x^0) + o(t^{2d(q)}) \geq c \sum_{\alpha \in V_\Omega} x^\alpha, \quad \text{for } t \text{ sufficiently large.}$$

On the other hand, each point of V_Ω has even coordinates, this inequality implies that

$$(F_g)_{\Delta'}(x^0) > 0.$$

By Claim 3.6,

$$(F_g)_{\Delta'}(x^0) = \sum_{i=1}^m g_{i,\Delta}^2(x^0) > 0.$$

It follows that,

$$\max |g_{i,\Delta}(x^0)| \neq 0.$$

□

Claim 3.8. A_Ω is an open subset of N_Ω .

PROOF. Assume $f(x) = \sum c_\alpha x^\alpha \in A_\Omega$. Therefore, there exist $r > 0$, $c > 0$ such that

$$\|x\| \geq r \implies f(x) \geq c \sum_{\alpha \in V_\Omega} x^\alpha.$$

We shall show that there exists a number $\epsilon > 0$ such that, if $|\delta_\alpha| < \epsilon$, for any $\alpha \in \Omega$, then $\tilde{f}(x) := \sum_{\alpha \in \Omega} (c_\alpha + \delta_\alpha) x^\alpha \in A_\Omega$.

In fact, if $\|x\| \geq r$ then

$$(3.12) \quad \tilde{f}(x) \geq \sum_{\alpha \in \Omega} c_\alpha x^\alpha - \sum_{\alpha \in \Omega} |\delta_\alpha| |x|^\alpha.$$

By [7] (Lemma 1, p. 160), if $\alpha \in \Omega \cap \mathbb{N}^n$ then

$$(3.13) \quad |x|^\alpha \leq \sum_{\alpha \in V_\Omega} x^\alpha.$$

Thus

$$(3.14) \quad \sum_{\alpha \in \Omega \cap \mathbb{N}^n} |\delta_\alpha| |x|^\alpha \leq \sum_{\alpha \in \Omega \cap \mathbb{N}^n} \epsilon |x|^\alpha \leq \epsilon \eta \sum_{\alpha \in V_\Omega} x^\alpha,$$

where η is the number of integer points in Ω .

Combining the inequalities (3.12), (3.13), and (3.14) we get the following inequality

$$\tilde{f}(x) \geq (c - \epsilon \eta) \sum_{\alpha \in V_\Omega} x^\alpha,$$

for all $x \in \mathbb{R}^n$ satisfying $\|x\| \geq r$.

Thus, if $\epsilon = \frac{c}{2\eta}$ then $\tilde{f}(x) \geq \frac{c}{2} \sum_{\alpha \in V_\Omega} x^\alpha$. Therefore $\tilde{f}(x) \in A_\Omega$ and the claim 3.8 is proved. \square

We continue the proof of Theorem 2.6.

Assume that $g_0 \in \mathcal{D}_\Gamma$. We will show that there exists an open neighborhood $U(g_0)$, s.t. $U(g_0) \subset \mathcal{D}_\Gamma$. By the claim 3.7, since $g_0 \in \mathcal{D}_\Gamma$ we have

$$F(g_0) \in A_\Omega.$$

By Claim 3.8, there exist an open set $V \subset A_\Omega$, containing $F(g_0)$. The mapping $F : \mathcal{N}_\Gamma \longrightarrow N_\Omega$ is continuous, then there exists an open neighborhood $U(g_0)$ of g_0 , such that

$$F(U(g_0)) \subset V.$$

Takes any element $g \in U(g_0)$, we have $F(g) \in V$. Hence $F(g) \in A_\Omega$. Now, it follows from Claim 3.7, $g \in \mathcal{D}_\Gamma$. Thus the open set $U(g_0)$ is contained \mathcal{D}_Γ . \square

Acknowledgments.

We would like to thank Professor Lê Dung Trang, who provided insight and expertise that greatly assisted this research.

This paper is supported by Vietnams National Foundation for Science and Technology Development (NAFOSTED).

REFERENCES

- [1] V. Arnold, S. Gusein-Zade, A. Varchenko, *Singularities of Differentiable Maps, vol. II*, Birkhäuser, Basel, 1988.
- [2] A. Barvinok, *A course in convexity*, American Mathematical Society, 2002.
- [3] A. Brøndsted, *An Introduction to Convex Polytopes*, Springer-Verlag, 1983.
- [4] Dinh Dung, *Number of Integral Points in a Certain Set and the Approximation of Functions of Several Variables*. Matematicheskije Zemarki, Vol. 36, No. 4 (1984), p. 479-491.
- [5] J. Denef, J. Nicaise, P. Sargos, *Oscillating integrals and Newton polyhedra.*, J. Anal. Math. 95 (2005), 147-172.
- [6] S.G. Gindikin, *Energy estimates connected with the Newton polyhedron*, (Russian) Trudy Moskovskogo Matematicheskogo Obshchestva, 31, 189-236. (English) Trans. Moscow Math. Soc., 31 (1974), 193-246.
- [7] S. Gindikin, L.R. Volevich, *The Method of Newton's Polyhedron in the Theory of Partial Differential Equations*, Kluwer Academic Publishers, 1992.
- [8] M. Greenblatt, *Resolution of singularities in two dimensions and the stability of integrals*, Advances in Mathematics 226 (2011), 17721802.
- [9] M. Greenblatt, *Stability of Oscillatory Integral Asymptotics in Two Dimensions*, J. Geom. Anal., (2012), p.1-28.
- [10] B. Grünbaum, *Convex Polytopes*. Springer, 2003.
- [11] A.I. Karol', *Newton Polyhedra, Asymptotics of Volumes, and Asymptotics of Exponential Integrals*, Amer. Math. Soc. Transl. (2) Vol. 228, 2009.
- [12] V.N. Karpushkin, *Uniform estimates of oscillatory integrals with parabolic or hyperbolic phases*, J. Soviet Math. 33 (1986), 11591188.
- [13] V.N. Karpushkin, *Uniform estimates for volumes*, Tr. Mat. Inst. Steklova, 221 (1998), 225231
- [14] A.G. Kouchnirenko, *Polydres de Newton et nombres de Milnor*, Invent. Math. 32 (1976), p. 1-31.
- [15] V. P. Mikhailov, *The behaviour at infinity of a class of polynomials*, Trudy Mat. Inst. Steklov. 91 (1967), 59-80. (Russian).
- [16] D.H. Phong, E.M. Stein, *The Newton polyhedron and oscillatory integral operators*, Acta Math., 179 (1997), 105-152.
- [17] D.H. Phong, E.M. Stein, J.A. Sturm, *On the growth and stability of real-analytic functions*. Am. J. Math. 121(3), 519554 (1999).
- [18] D.H. Phong, E.M. Stein, J.A. Sturm, *Multilinear level set operators, oscillatory integral operators, and Newton polyhedra*, Math. Ann. 319 (2001), 573-596 .
- [19] A. Seeger, *Radon transforms and finite type conditions*, J. of the AMS. Vol. 11, N. 4, 1998, p. 869-897.
- [20] E.V. Sinitskaya, *Newton's Polyhedron and Weyl's Formula for the spectrum of the Schrödinger operator with polynomial potential*, Journal of Mathematical Sciences, Vol. 124, No. 3 (2004), 5036-5053.
- [21] A.N. Varchenko, *Newton polyhedra and estimations of oscillatory integrals*, Functional Anal. Appl. 10 (1976), 175196.
- [22] V.A. Vasiliev, *Asymptotic Exponential Integrals, Newton's Diagram, and the Classification of Minimal Points*. Moscow State University. Vol. 11, No. 3 (1977), p. 1-11.
- [23] V.A. Vasiliev, *Asymptotic behavior of exponential Integrals in the complex domain*. Funkt. Anal. Prilozh. 18 (1979), 239-247.
- [24] Ziegler G.M., *Lectures on Polytopes*. Springer, 1995.

INSTITUTE OF MATHEMATICS, VIETNAM ACADEMY OF SCIENCES, HANOI, VIETNAM, 18, HOANG QUOC VIET, CAU GIAY, HANOI

E-mail address: hhvui@math.ac.vn

TEACHER TRAINING COLLEGE OF DALAT, 29 YERSIN, DALAT, VIETNAM

E-mail address: gialoc@gmail.com